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Zakharov–Shabat Inverse Problem with Meromorphic Reflection Coefficient

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We analyze the Zakharov–Shabat-type inverse problem when the reflection coefficient contains poles in the eigenvalue plane, as an extension of the earlier work by Atkinson in the case of the Schrödinger problem. It is demonstrated that due to the mutual influence of such a pole and the usual bound-state pole, a discontinuous solitary wave profile is generated. Furthermore, we also examine the form of the nonlinear field only due to the pole of the reflection coefficient. A different approach is necessary to convert the GLM equation into a purely differential one for its solution.

1. INTRODUCTION

The inverse scattering transform (IST) is one of the best approaches for the solution of nonlinear integrable systems. Of late various other methodologies have also been suggested, but the flexibility of IST is perhaps the best. In recent years some authors have examined the inverse problem for the Schrödinger equation (Levi and Ragnisco, 1985; Pechenick and Cohen, 1981) when the reflection coefficient $R(K)$ contains poles in the complex eigenvalue plane (Lamb, 1980). It may be mentioned that the corresponding Gelfand–Levitan equation requires a different procedure for its solution. For the case of the KdV equation such a problem has been analyzed by Atkinson (1988). In this communication we study the Zakharov–Shabat-type inverse problem, when the reflection coefficient has a pole in the eigenvalue plane. Two cases are investigated—one in the presence of the usual bound-state pole, the other when it is absent. In the first case we observe that the soliton-like profile is generated with a discontinuity, while in the latter case no soliton solution can be seen.

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2. FORMULATION

The Zakharov–Shabat (1971)-type eigenvalue problem is written as (Ablowitz *et al.*, 1974)

$$\frac{\partial \Psi}{\partial x} = \begin{pmatrix} i\lambda & q \\ r & -i\lambda \end{pmatrix} \Psi \tag{1}$$

where (q, r) are the nonlinear field variables and Ψ is a two-component linear eigenfunction, $\Psi = (\Psi_1, \Psi_2)^+$. The required inverse scattering formalism is already formulated in Ablowitz *et al.* (1974), so we adopt their notations. We assume the usual asymptotic vanishing condition for q and r. Let $(\phi, \bar{\phi})$ and $\Psi, \bar{\Psi}$ denote the Jost functions for $x \to -\infty$ and $x \to +\infty$, respectively.

Then

$$\begin{aligned} \phi &= a(\lambda)\bar{\Psi} + b(\lambda)\Psi \\ \bar{\phi} &= -\bar{a}(\lambda)\Psi + \bar{b}(\lambda)\bar{\Psi} \end{aligned} \tag{2}$$

$(a, b, \bar{a}, \bar{b})$ represent the scattering data. The usual analyticity property of these are also assumed to hold good. The kernel of the Gelfand–Levitan equation is written as

$$M(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{a} e^{i\lambda x}\, d\lambda - i \sum_j C_j e^{i\lambda x} \tag{3}$$

The sum over the discrete terms in equation (3) denotes the contribution from the bound-state poles, and $b/a = R(x)$ and $\bar{b}/\bar{a} = \bar{R}(x)$ are the reflection coefficients. We now assume that the reflection coefficients are not zero, but each has a pole on the imaginary axis. That is,

$$\begin{aligned} R(\lambda) &= \frac{r_1}{\lambda - i\beta_1} \\ \bar{R}(\lambda) &= \frac{r_2}{\lambda - i\beta_2} \end{aligned} \tag{4}$$

Then the contribution from the continuous part in equation (3) can be extracted with the help of a contour integral and we get (with the assumption of a single bound-state pole)

$$\left.\begin{aligned} M_+(x) &= -\alpha_1 e^{-\beta_1 x}\theta(\beta_1) + \gamma_1 e^{-\delta_1 x} \\ M_-(x) &= \alpha_1 e^{\beta_1 x}\theta(-\beta_1) + \gamma_1 e^{-\delta_1 x} \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} \bar{M}_+(x) &= -\alpha_2 e^{-\beta_2 x}\theta(\beta_2) + \gamma_2 e^{-\delta_2 x} \\ \bar{M}_-(x) &= \alpha_2 e^{\beta_2 x}\theta(-\beta_2) + \gamma_2 e^{-\delta_2 x} \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} M &= M_+(x)\theta(x) + M_-(x)\theta(-x) \\ \bar{M} &= \bar{M}_+(x)\theta(x) + \bar{M}_-(x)\theta(-x) \end{aligned}\right\} \tag{7}$$

where α_i, β_i, γ_i, and δ_i are arbitrary constants to be specified later.

Since the general situation is extremely complicated, we concentrate on a special case, but retaining the salient features. The two functions $K(x, y)$ and $\bar{K}(x, y)$ which solve the inverse problem for q and r are solutions of the following two integral equations (Ablowitz, 1978):

$$K(x, y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{M}(x + y) - \int_x^\infty dz\, \bar{K}(x, z)\bar{M}(z + y) = 0 \tag{8}$$

$$\bar{K}(x, y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} M(x + y) + \int_x^\infty dz\, K(x, z)M(z + y) = 0 \tag{9}$$

Since the kernels M and $\bar{M}$ have different structures for $x > 0$ and $x < 0$, we are to subdivide the range $[x, \infty]$ accordingly. Lastly we mention that q and r are reproduced through

$$\begin{aligned} q(x) &= -2K_1(x, x) \\ r(x) &= -2\bar{K}_2(x, x) \end{aligned} \tag{10}$$

In the following we show how K_1 and $\bar{K}_2$ can be obtained explicitly with M and $\bar{M}$ given in equations (5) and (6).

3. METHOD OF SOLUTION

Let us start by choosing the three regions $[x > 0; x + y > 0]$, $[x < 0, x + y > 0]$, and $[x < 0, x + y < 0]$ separately. Also set

$$\begin{aligned} &\left.\begin{aligned} K &= P \\ \bar{K} &= \bar{P} \end{aligned}\right\} \quad \text{when } x > 0, \quad x + y > 0 \\ &\left.\begin{aligned} K &= Q \\ \bar{K} &= \bar{Q} \end{aligned}\right\} \quad \text{when } x < 0, \quad x + y > 0 \\ &\left.\begin{aligned} K &= T \\ \bar{K} &= \bar{T} \end{aligned}\right\} \quad \text{when } x < 0, \quad x + y < 0 \end{aligned} \tag{11}$$

Then the GLM equations (6) and (9) break up into the following set of coupled integral equations:

$$P(x, y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{M}_+(x + y) - \int_x^\infty dz\, \bar{P}(x, z)\bar{M}_+(z + y) = 0 \tag{12}$$

$$Q(x, y) - \begin{pmatrix}1\\0\end{pmatrix}\bar{M}_+(x+y) - \int_{-x}^{\infty} dz\, \bar{Q}(x, z)\bar{M}_+(z+y) - \int_{x}^{-x} dz\, \bar{T}(x, z)\bar{M}_+(z+y) = 0 \tag{13}$$

$$T(x, y) - \begin{pmatrix}1\\0\end{pmatrix}\bar{M}_-(x+y) - \int_{-x}^{\infty} dz\, \bar{Q}(x, z)\bar{M}_+(z+y) - \int_{-y}^{-x} dz\, \bar{T}(x, z)\bar{M}_+(z+y) - \int_{x}^{-y} dz\, \bar{T}(x, z)\bar{M}_-(z+y) = 0 \tag{14}$$

along with the following for P, $\bar{Q}$, and $\bar{T}$:

$$\bar{P}(x, y) + \begin{pmatrix}0\\1\end{pmatrix}M_+(x+y) + \int_{x}^{\infty} dz\, P(x, z)M_+(z+y) = 0 \tag{15}$$

$$\bar{Q}(x, y) + \begin{pmatrix}0\\1\end{pmatrix}M_+(x+y) + \int_{-x}^{\infty} dz\, Q(x, z)M_+(z+y) + \int_{x}^{-x} dz\, T(x, z)M_+(z+y) = 0 \tag{16}$$

$$\bar{T}(x, y) + \begin{pmatrix}0\\1\end{pmatrix}M_-(x+y) + \int_{-x}^{\infty} dz\, Q(x, z)M_+(z+y) + \int_{-y}^{-x} dz\, T(x, z)M_+(z+y) + \int_{x}^{-y} dz\, T(x, z)M_-(z+y) = 0 \tag{17}$$

whence the expression for q and v can be written as

$$q(x) = -2[P_1(x, x)\theta(x) + T_1(x, x)\theta(-x)] \tag{18}$$

$$r(x) = 2[\bar{P}_2(x, x)\theta(x) + \bar{T}_2(x, x)\theta(-x)] \tag{19}$$

In the above we have always used the notation that each of P, Q, and T is a two component vector written as $(P_1, P_2)^t$, $(Q_1, Q_2)^t$, $(T_1, T_2)^t$, etc. So each of the above equations (14)–(16) breaks up into two coupled equations for their component functions. To avoid the clumsy nature of the final expression, we set

$$\alpha_1 = \beta_1 = \delta_1 > 0; \qquad \alpha_2 = \beta_2 = \delta_2 > 0$$

It may be noted that equation (12) for P can be solved in the usual manner and we get

$$P_1(x, x) = \frac{(\gamma_2 - \alpha_2)e^{-2\alpha_2 x}}{1 + [(\gamma_1 - \alpha_1)(\gamma_2 - \alpha_2)/(\alpha_1 + \alpha_2)^2]e^{2(\alpha_1 + \alpha_2)x}} \tag{20}$$

However, for the equations for T_1 we note that the y dependence is not as in the usual GLM equation, because of the finite limits occurring in the integrals. So we set $T_1(x, y) = t_1(x, y)e^{-\alpha_2 y}$ and $\bar{T}_1(x, y) = \bar{t}_1(x, y)e^{-\alpha_1 y}$.

Such equations upon differentiation yield

$$\begin{aligned} \bar{t}_1(x, -y) &= -\frac{1}{\alpha_2} e^{-(\alpha_1+\alpha_2)y} t_1'(x, y) \\ t_1(x, -y) &= \frac{1}{\alpha_1} e^{-(\alpha_1+\alpha_2)y} \bar{t}_1'(x, y) \end{aligned} \tag{21}$$

Again changing y to $-y$ and differentiating, we obtain

$$\begin{aligned} t_1'(x, -y) &= \alpha_2 e^{-(\alpha_1+\alpha_2)y} \bar{t}_1(x, y) \\ \bar{t}_1'(x, -y) &= -\alpha_1 e^{-(\alpha_1+\alpha_2)y} t_1(x, y) \end{aligned} \tag{22}$$

Using these relations, we arrive at a second-order differential equation

$$t_i''(x, y) - (\alpha_1 + \alpha_2) t_i'(x, y) - \alpha_1 \alpha_2 t_i(xy) = 0 \tag{23}$$

for $i = 1, 2$, whence we get

$$\begin{aligned} t_1(x, y) &= [A(x)e^{ny} + B(x)e^{-ny}] e^{[(\alpha_1+\alpha_2)/2]y} \\ n &= \tfrac{1}{2}[(\alpha_1 + \alpha_2)^2 + 4\alpha_1\alpha_2]^{1/2} \end{aligned} \tag{24}$$

Similarly,

$$\bar{t}_1(x, y) = [\bar{A}(x)e^{ny} + \bar{B}(x)e^{-ny}] e^{[(\alpha_1+\alpha_2)/2]y} \tag{25}$$

To determine A, $\bar{A}$ and B, $\bar{B}$ we set $Q_1(x, y) = q_1(x)e^{-\alpha_2 y}$ and $Q_1(x, y) = \bar{q}_1(x)e^{-\alpha_1 y}$, to get a set of coupled linear equations for these which can be easily solved. Finally we obtain,

$$-q(x) = 2[P_1(x, x)\theta(x) + T_1(x, x)\theta(-x)] \tag{26}$$

with P_1 given in equation (24), T_1 in (25), along with

$$\begin{aligned} A(x) &= \frac{C_1 b_2 - C_2 b_1}{a_1 b_2 - a_2 b_1} \\ B(x) &= \frac{C_1 a_2 - C_2 a_1}{b_1 a_2 - b_2 a_1} \end{aligned} \tag{27}$$

where

$$\begin{aligned} a_1 &= \left[\frac{\gamma_1}{n_1}(e^{n_2 x} - e^{-n_2 x}) - \frac{\alpha_1}{n_2} e^{n_2 x}\right] \\ &\quad - \frac{a_1^n}{a_1^d}\left[\frac{(\gamma_1 - \alpha_1)}{n_2(\alpha_1 + \alpha_2)}(e^{(\alpha_1+\alpha_2+n_2)x} - e^{n_1 x}) - \frac{\alpha_1}{n_1 n_2}(e^{n_1 x} - e^{-n_1 x})\right] \\ a_1^n &= \frac{(\gamma_1 - \alpha_1)(\gamma_2 - \alpha_2)}{(\alpha_1 + \alpha_2)} e^{(\alpha_1 + \alpha)x} \\ a_1^d &= 1 + \frac{(\gamma_1 - \alpha_1)(\gamma_2 - \alpha_2)}{(\alpha_1 + \alpha_2)^2} e^{2(\alpha_1+\alpha_2)x} \end{aligned} \tag{28}$$

Similar expressions hold good for other coefficients b_1, C_1, C_2, b_2, etc. We do not reproduce these here, because they do not convey any physical information. We plot the expression (26) numerically. We have used three sets of values for the parameters occurring in the expression (26). These are $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0.25$, $\gamma_2 = 0.75$; $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0.75$, $\gamma_2 = 0.25$; and $\gamma_1 = 1.0$, $\gamma_2 = 1.0$ with α_1, α_2 as before.

4. DISCUSSION

It is interesting to observe that in the first case when we include the bound-state pole, we reproduce a solitary wave-like profile with a discontinuity on the ordinate. Two such cases have been reproduced in Figs. 1–3. On the other hand, when we exclude the bound-state pole, the profile of the generated wave looks like the objects shown in Fig. 4. This particular form is in no way related to the solitary wave solution. Of course there are nonlinear equations where nonreflectionless solitons are known to occur. The whole process of calculation can be adopted even for the N-soliton case with m number of poles of $R(\lambda)$. But the computation will be terribly complicated.

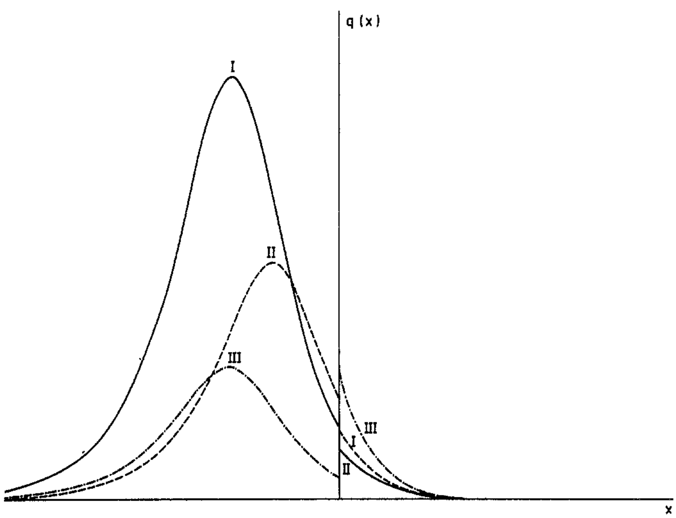

Fig. 1. Plot of $q(x)$ versus x in the presence of bound-state pole. (I) $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0.25$, $\gamma_2 = 0.75$, (II) $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0.75$, $\gamma_2 = 0.25$, (III) $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 1.0$, $\gamma_2 = 1.0$.

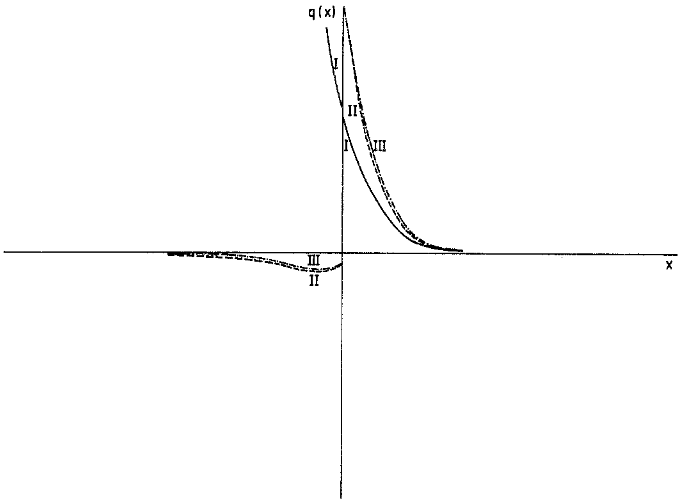

Fig. 2. Plot of $q(x)$ versus x in the absence of bound-state pole. (I) $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0$, $\gamma_2 = 0.75$, (II) $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0.25$, $\gamma_2 = 0$, (III) $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\gamma_1 = 0$, $\gamma_2 = 0$.

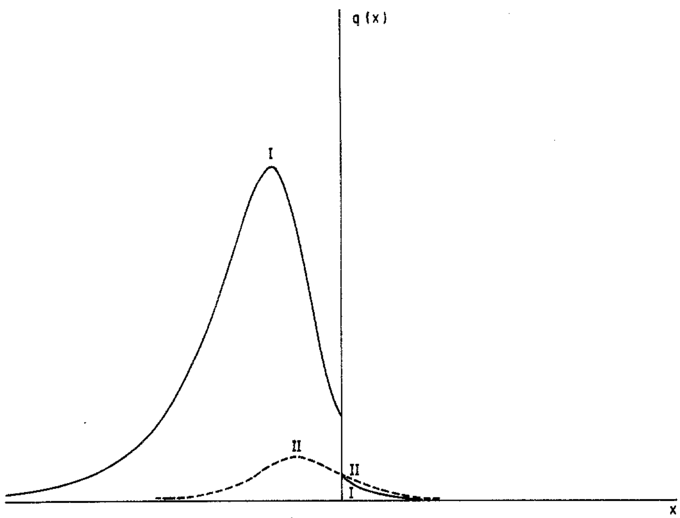

Fig. 3. Plot of $q(x)$ versus x in the presence of bound-state pole. (I) $\alpha_1 = 1$, $\alpha_2 = 2$, $\gamma_1 = 0.5$, $\gamma_2 = 1.5$, (II) $\alpha_1 = 2$, $\alpha_2 = 1$, $\gamma_1 = 1.5$, $\gamma_2 = 0.5$.

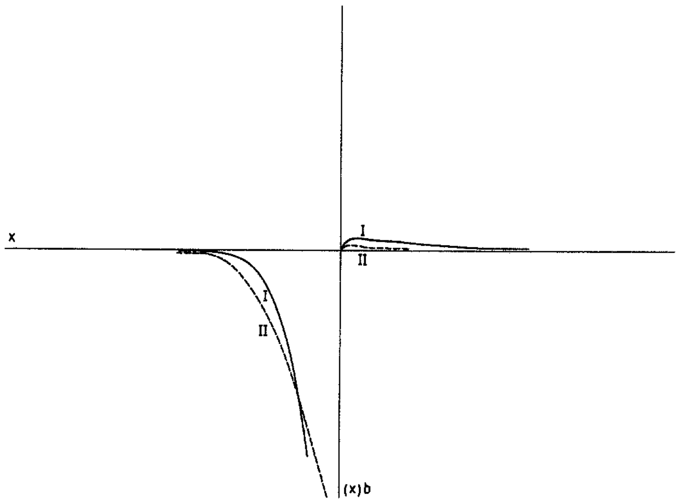

Fig. 4. Plot of $q(x)$ versus x in the absence of bound-state pole. (I) $\alpha_1 = 1$, $\alpha_2 = 2$, $\gamma_1 = 0.5$, $\gamma_2 = 0$, (II) $\alpha_1 = 2$, $\alpha_2 = 1$, $\gamma_1 = 1.5$, $\gamma_2 = 0$.

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